The Velocity Increment for Hohmann Coplanar Transfer from Different Low Earth Orbits

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Abstract

The transfer of satellites in too high orbits as geosynchronous one (geostationary), usually is achieved firstly by launching the satellite in Low Earth Orbit (LEO), then in elliptical transfer orbit and finally to the geosynchronous orbit. The three steps process is known as Hohmann transfer. The Hohmann transfer which involves two circular orbits with different orbital inclinations is known as non-coplanar Hohmann transfer. If both orbital planes are aligned the Hohmann transfer is known as coplanar what is further considered in this paper. In terms of propellant consumptions the Hohmann transfer is the best known transfer to be applied when transferring between circular coplanar orbits. For transfer between circular coplanar orbits, the given information usually consists of the radii of the initial and final orbits. The velocity to be applied into two orbit points in order to attain the dedicated final orbit is analyzed. The aim of this paper is to conclude about the velocity changes under the different initial low Earth altitudes. For different initial orbit altitudes, the velocities to be applied in process of Hohmann transfer are simulated. From respective simulations, the velocity variations on dependence of initial altitudes are derived. The eccentricity is considered, too.

Keywords

LEO; Satellite; Orbit; Hohmann Transfer

Introduction

When launching and then consolidating the satellite on its own high circular orbit (ex. geosynchronous), the needed satellite’s propellant mass must be minimized. The Hohmann transfer is well known for the minimum of propellant mass used for satellite transfer into high orbits.

The Hohmann transfer is the best transfer to be used when transferring between circular coplanar orbits [1]. For transfer between circular coplanar orbits, the information usually given consists of the radii of the initial and final orbits. Due to the reversibility of orbits, Hohmann transfer orbits also work to bring a spacecraft from the higher orbit into the lower one. The Hohmann transfer orbit is based on two instantaneous velocity changes. The transfer consists of a velocity impulse on an initial circular orbit, in the direction of motion and collinear with velocity vector, which propels the space vehicle into an elliptical transfer orbit. The second velocity impulse also in the direction of motion is applied at apogee of the transfer orbit which propels the space vehicle into a final circular orbit at the final altitude [1, 2].

This paper is motivated by the fact that the problem of Hohmann transfer by [3] is analyzed with three normalized radii, and our approach is based on the single normalized radius for the coplanar transfer case. The non coplanar transfer involves also allocating the total plane change angle between two maneuvers [2].

Under the first section the elliptic orbit is generally considered. Further, through implementation of the single normalized radius, both velocity impulses to be applied at perigee and apogee of the Hohmann transfer orbit are analyzed. Considering different initial low Earth orbit altitudes respective velocities are calculated. Finally, these results are discussed and closed by conclusions.
Elliptic Orbit

The path of the satellite’s motion is an orbit. Generally, the orbits of communication satellites are ellipses laid on the orbital plane defined by space orbital parameters. These parameters (Kepler elements) determine the position of the orbital plane in space, the location of the orbit within orbital plane and finally the position of the satellite in the appropriate orbit [4,5]. The exactly know position of the satellite in space enables the communication between the satellite and ground stations (users) [6]. For reliable communication, during the link budget calculations atmospheric impairments and interference aspects have carefully to be considered [7, 8]. The communication between the satellite and a ground station is established only when the satellite is stabilized in its own orbit. Thus, permanent attitude control is mandatory. In terms of attitude control performance the satellite reaction wheel’s configuration also plays an important role in providing the attitude control torques [9]. Different algorithms are applied and active control means are generally added to assure accurate attitude stabilization, keeping the attitude errors within permitted limits, consequently keeping the reliable communication [10, 11].

The elliptic orbit is determined by the semi-major axis which defines the size of an orbit, and the eccentricity which defines the orbit’s shape. Orbits with no eccentricity are known as circular orbits. The elliptic orbit shaped as an ellipse, with a maximum extension from the Earth center at the apogee (r_a) and the minimum at the perigee (r_p) is presented in Figure 1.

![FIGURE 1. MAJOR PARAMETERS OF AN ELLIPTIC ORBIT.](image)

The orbit’s eccentricity is defined as the ratio of difference to sum of apogee (r_a) and perigee (r_p) radii as, [4]-[5].

\[ e = \frac{r_a - r_p}{r_a + r_p} \]  

(1)

Applying geometrical features of ellipse yield out the relations between semi major axis, apogee and perigee:

\[ r_p = a(1 - e) \]  

(2)

\[ r_a = a(1 + e) \]  

(3)

\[ 2a = r_a + r_p \]  

(4)

both, \( r_p \) and \( r_a \) are considered from the Earth’s center. Earth’s radius is \( r_E = 6378 \) km. Then, the altitudes (highs) of perigee and apogee are:

\[ h_p = r_p - r_E \]  

(5)

\[ h_a = r_a - r_E \]  

(6)

Different methods are applied for satellite injection missions. Goal of these methods is to manage and control the satellite to safely reach the low Earth orbit, and then to the transfer elliptical orbit and finally the geosynchronous orbit [12]-[14]. The Hohmann transfer is considered as the most convenient.

The specific orbit implementation depends on satellite’s injection velocity. The orbit implementation process on the
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best way is described in terms of the cosmic velocities. Based on Kepler’s laws, considering an elliptic orbit, the satellite’s velocity at the perigee and apogee point, respectively are expressed as [4, 5],

\[ v_p = \sqrt{\frac{2\mu}{r_p}} - \frac{2\mu}{r_p + r_p} \]  \( (7) \)

\[ v_a = \sqrt{\frac{2\mu}{r_a}} - \frac{2\mu}{r_a + r_p} \]  \( (8) \)

\[ v_p \cdot r_p = v_a \cdot r_a = vrcos\Phi \]  \( (9) \)

\[ \mu = m \cdot G = 3.986 \cdot 10^5 \text{ km}^3 / \text{s}^2, \]  \[ G \] is the Earth’s gravitational constant and \( m \) is Earth’s mass, \( \Phi \) represents an angle between a satellite vector \( \mathbf{r} \) and local horizon at satellite point. For orbit with no eccentricity \( (e = 0) \), apogee and perigee distances are equal \( (r_a = r_p = r) \), thus orbit becomes circular with radius \( r \) and orbital velocity as [2]:

\[ v_1 = \sqrt{\frac{\mu}{r}} \]  \( (10) \)

By definition this is called the first cosmic velocity, enabling the satellite to orbit circularly around the Earth at the uniform velocity according to (10). If the injection velocity happens to be less than the first cosmic velocity, the satellite follows a ballistic trajectory and falls back to the Earth [4, 5]. The second cosmic velocity is expressed as,

\[ v_2 = \sqrt{\frac{2\mu}{r}} \]  \( (11) \)

A spacecraft under second cosmic velocity leaves the Earth’s gravity. For injection velocity \( v_p \) at perigee more than the first cosmic velocity and less than the second cosmic velocity the orbit is elliptical with an eccentricity in between 0 and 1. This is expressed as:

\[ v_1 < v < v_2 \]  \( (12) \)

\[ 0 < e < 1 \]  \( (13) \)

The satellite injection point is at perigee, and the apogee distance attained in the elliptical orbit depends upon the injection velocity. The higher is the injection velocity at perigee, the greater is the apogee distance. For the same perigee distance \( r_p \), if under the injection velocity \( v_{pl} \) at perigee it is attained an apogee distance \( r_{a1} \), and under velocity \( v_{p2} \) it is attained an apogee distance \( r_{a2} \), then applying (7) yields out the relationship between velocities at perigee and respective attained distances at apogee.

\[ \left( \frac{v_{p2}^2}{v_{pl}} \right)^2 = \frac{1 + \frac{r_p}{r_{a1}}}{1 + \frac{r_p}{r_{a2}}} \]  \( (14) \)

Coplanar Hohmann Transfer

The Hohmann coplanar transfer orbit is an elliptic orbit used to transfer between two circular orbits of different radii in the same plane. The first (low radius) circular orbit is defined as initial and the second (high radius) is defined as a final orbit. The orbital maneuver to perform the Hohmann transfer applies two engine impulses (thrusts), one to move a space craft onto the transfer orbit and a second to move off it. For the coplanar Hohmann transfer, two applied velocity impulses are confined to the orbital planes of the initial and final orbits.

The transfer from the low radius orbit to the high radius orbit is attained in three steps. The first one is the launch of the satellite in the low Earth circular orbit (LEO). This is the initial circular orbit with radius of \( r \) as it is presented in Figure 2. By the second step, the first velocity impulse (\( \Delta v_1 \)) is applied at the low Earth orbit creating an elliptic transfer orbit with perigee altitude equal of the initial circular orbit and the apogee altitude equal to the
final circular orbit. Finally, at the third step, the second velocity impulse \( \Delta v_2 \) is applied at apogee of the transfer orbit in order to attain the final orbit, respectively the geosynchronous circular orbit (GEO) with radius \( r_f \) as presented in Figure 2. The apogee of the transfer orbit is equal to the radius of the final orbit. Thus, the second velocity impulse circularizes the transfer orbit at apogee. Both velocity impulses \( \Delta v_1 \) and \( \Delta v_2 \) keep the direction of the orbits' motion.

\[
2a_n = r_f + r_m
\]

Based on (10) will have the velocity of initial circular orbit as \( v_m \) and the velocity of final circular orbit as \( v_f \). Thus, \( v_m \) depends on the LEO radius \( r_m \), and \( v_f \) remains unchanged because of \( r_f \) unchangebility.

\[
v_m = \sqrt{\frac{\mu}{r_m}} \tag{16}
\]

\[
v_f = \sqrt{\frac{\mu}{r_f}} \tag{17}
\]

For elliptic orbit with perigee equal to \( r_m \) and apogee of \( r_f \) the velocities at perigee and apogee are:

\[
v_{pm} = \sqrt{\frac{2\mu}{r_m} - \frac{2\mu}{r_m + r_f}} \tag{18}
\]

\[
v_{am} = \sqrt{\frac{2\mu}{r_f} - \frac{2\mu}{r_m + r_f}} \tag{19}
\]

\( v_{pm} \) is in fact the velocity in the transfer orbit at initial orbit height and \( v_{am} \) in fact is the velocity in the transfer orbit at final orbit height. The initial velocity increment \( \Delta v_{in} \), to move the satellite from the initial circular orbit to the elliptic transfer orbit, is given as the difference between velocity at perigee of transfer orbit \( v_{pm} \) and the velocity of
the initial circular orbit $v_i$ and the as:

$$\Delta v_i = v_m - v_i$$  \hspace{1cm} (20)

The final velocity increment ($\Delta v_{2a}$), to move the satellite from elliptic transfer orbit to geosynchronous circular orbit, is given as the difference between the velocity on the final circular $v_f$ orbit and the velocity on the apogee of the transfer elliptical orbit $v_{in}$.

$$\Delta v_{2a} = v_f - v_{in}$$  \hspace{1cm} (21)

Applying (18) to (21) will have,

$$\Delta v_{2a} = \mu \sqrt{\frac{2r_f}{r_m + r_f} - 1}$$  \hspace{1cm} (22)

$$\Delta v_{2a} = \mu \sqrt{\frac{1 - \sqrt{2r_f}}{r_m + r_f}}$$  \hspace{1cm} (23)

Further it is defined normalized radius $R_n$ as follows,

$$R_n = \frac{r_m}{r_f}$$  \hspace{1cm} (24)

from which one yields out

$$v_f = R_n \cdot v_{in}$$  \hspace{1cm} (25)

Applying (24) and (25) at (22) and (23), finally will have velocity increments to be applied at Hohmann coplanar transfer orbit in order to attain geosynchronous orbit from different low Earth orbits, as:

$$\Delta v_{in} = v_{in} \cdot \left(\sqrt{\frac{2}{R_n^2 + 1}} - 1\right)$$  \hspace{1cm} (26)

$$\Delta v_{2a} = v_{in} \cdot R_n \cdot \left(1 - R_n \cdot \sqrt{\frac{2}{R_n^2 + 1}}\right)$$  \hspace{1cm} (27)

From the propellant consumption point of view, it of interest the contribution of both velocity impulses as:

$$\Delta v = \Delta v_{in} + \Delta v_{2a}$$  \hspace{1cm} (28)

The eccentricity of the transfer orbit for different low Earth orbits is given as,

$$e_n = \frac{1 - R_n^2}{1 + R_n^2}$$  \hspace{1cm} (29)

**Calculations and Results**

The transfer is initiated by firing the space craft engine at low Earth orbit in order to accelerate it so that it will follow the elliptical orbit; this adds energy to the space craft’s orbit. When the spacecraft has reached transfer orbit, its orbital speed (and hence its orbital energy) must be increased again in order to change the elliptic orbit to the larger circular one, respectively to geosynchronous one.

For simulation purposes five altitudes of low Earth orbits are considered as initial orbits for the Hohmann transfer, starting from altitude of 500km up to 1300km which are typical for LEO satellites. These altitudes correspond to initial radii of 6878km up to 7678km. For each of them it is calculated ($\Delta v_i$) and ($\Delta v_{2a}$) additionally ($\Delta v$). These results are presented in Table 1 and in Figure 3.
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**TABLE 1. VELOCITY IMPULSES TO BE APPLIED AT ELLIPTIC HOHMANN TRANSFER ORBIT.**

Figure 3 confirms that as higher is the altitude of the low initial orbit, the lower velocity impulse is needed to the final destination orbit. Consequently less fuel is needed to be carried out on the satellite in case that the satellite initially is injected on the higher altitude toward the final geosynchronous destination.

**Conclusions**

In orbital mechanics, the Hohmann transfer orbit is an elliptical orbit used to transfer between two circular orbits of different radii. If both orbits lie in the same plane, it is known as coplanar transfer. The orbital maneuver to perform the Hohmann transfer applies two engine impulses (thrusts), one to move a space craft onto the transfer orbit and a second to move off it.

It is confirmed that as higher is the altitude of the low initial orbit, the lower velocity impulse is needed to the final destination orbit. Consequently less fuel is needed to be carried out on the space craft in case that the space craft initially is injected on the higher altitude.

Through simulation results, it is also confirmed that for different altitudes of initial orbit, the first velocity impulse has faster decreasing gradient than the second velocity impulse.

**REFERENCES**


Shkelzen Cakaj has received his BSc and MSc degrees from Prishtina University in Kosovo. Since 2003 he is cooperating with Institute for Communication and Radio – Frequency Engineering at the Technical University in Vienna, where he has prepared his Master Thesis related to the performance of the ground satellite station in July, 2004. He was awarded a PhD in area of satellite communication from Zagreb University in January 2008 with whom he has continued technical associations. He has attended courses on satellite communication and spectrum management at USTTI. He was awarded as Fulbright scholar researcher in 2009 at NOAA (National Oceanic and Atmospheric Administration) at Maryland, USA. He is the author of 50 papers published in worldwide conferences and journals; mostly IEEE. His area of interest is the performance of satellite ground stations for scientific satellites. He is working at Post and Telecommunication of Kosovo and lecturing satellite communications for master students at Prishtina University, Kosovo and Polytechnic University of Tirana, Albania.