Asymptotic Analysis of Thermoelastic Response in a Functionally Graded Solid Based on L-S Theory

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Abstract

This paper is concerned with the thermoelastic response in a functionally graded solid with an analytical method. The governing equations are proposed in the context of Lord-Shulman generalized theory (L-S theory). The Laplace transform techniques and some approximate treatments are employed to have an analytical solution for the thermoelastic response in a semi-infinite solid composed of functionally graded materials, whose boundary is subjected to a sudden thermal shock. Some important phenomena involving finite speed of heat signal are obtained. The comparison with the results for different values of non-homogeneous index is also conducted to evaluate the effect of graded material properties on thermoelastic response.

Keywords

Generalized Thermoelasticity; Functionally Graded Materials; Analytical Solution; Thermal Shock

Introduction

In recent years, functionally graded materials (FGMs), as a new class of intelligent material, has been widely applied in various engineering practices [1]. The analysis of thermoelastic response, especially the prediction of thermal stresses generated in severe temperature environments, is very important to evaluate the lives of FGMs. Many investigations have been conducted in the context of conventional theory of thermoelasticity, however, the conventional theory proposed by Biot [2], predicts an infinite speed of heat propagation, which contradicts physical facts and limits the applicability of these investigations to certain heat conduction. To eliminate this shortcoming, some modified theories admitting a finite propagation speed of heat signal are proposed on different perspective, which are also named as generalized theories of thermoelasticity. The widely applied theories include the L-S theory with one relaxation time [3], G-L theory with two relaxation time [4] and G-N theory with the assumption of no energy dissipation [5].

Within these generalized theories, some thermoelastic problems of FGMs with different boundary conditions have been studied by Bagri and Eslami [6], Darabseh et al. [7], Ghosh and Kanoria [8], where the power law distribution along the spatial position was used to describe the graded material properties. Meanwhile, the exponential distribution with spatial position for material properties was also employed to analyze the thermoelastic interaction in FGMs by Mallik and Kanoria [9], Tokovvy and Ma [10] respectively. Furthermore, Kanoria and Ghosh [11] studied the thermoelastic response further for a functionally graded spherically isotropic hollow, where the temperature-dependent properties were also considered. Abbas [12] also solved the thermoelastic interaction in a thick-walled FGM cylinder with the condition that material properties are the function of temperature and graded in the radial.

Since the complexity of the solution for governing equations with variable material properties, which are nonlinear forms in general cases, the theoretical treatment is very difficult and the closed-form solutions is rare in previous investigations [6-12]. Recently, an asymptotic approach [13], based on the Laplace transform and its limit theorem, has been introduced to solve some generalized thermoelastic problems [14, 15]. The asymptotic solutions with closed form can be obtained by this asymptotic approach, which is very effective to the problems involving short thermal duration, such as thermal shock, and is very convenient to evaluate thermo-mechanical properties of
FGMs in severe circumstances.

In this paper, the thermoelastic interaction in a functionally graded solid is studied with this asymptotic approach [13-15]. The governing equations of FGMs are derived in the context of L-S theory, where the exponentially distribution along the spatial position is used to describe the graded material properties. The closed-form solutions for a special problem of a semi-infinite functionally graded solid with the boundary subjected to a sudden thermal shock are derived. Utilizing these solutions, the propagation of each wave, the distributions of the displacement, temperature and stresses, and the variation of each distribution at different non-homogeneous index are obtained and plotted.

**Formulation of the Problem**

Due to the L-S theory, the governing equations of FGMs in absence of body forces and heat generation can be expressed as

\[
\sigma_{ij} = \lambda \gamma_{ik} \delta_{jk} + 2 \mu \gamma_{ij} - \beta \theta \delta_{ij},
\]

\[
\rho \ddot{u}_i = \sigma_{ij,i},
\]

\[
[K_{ij}] = \rho c_p \left( \tau_{ij} \tilde{T} + T_0 \beta \tau_{ij} \ddot{x} + \ddot{\gamma}_{ij} \right).
\]

where \( u_i \) are the components of the displacement vector, \( \sigma_{ij} \) are the components of the stress tensor, \( \gamma_{ij} = (u_{ij} + u_{ji})/2 \) are the components of the strain tensor, \( \theta = T - T_0 \) is the increment temperature, \( T \) is the absolute temperature, \( T_0 \) is the reference temperature, \( \rho \) is the mass density, \( k \) is the thermal conductivity, \( c_p \) is the specific heat at constant strain, \( \beta = (3\lambda + 2\mu)/\alpha \) is the thermal-mechanical coefficient, \( \alpha \) is the coefficient of linear thermal expansion, \( \lambda \) and \( \mu \) are the Lame’s constants, and \( \tau_0 \) is the thermal relaxation time constant defined in L-S theory. Meanwhile, the superscript dot (‘) and the subscript comma (,) denote the derivatives respect to the time \( t \) and coordinates \( x_i \) \((i = 1, 2, 3)\), respectively.

With the effects of functionally graded solid, the parameters \( \rho, \lambda, \mu, \beta, c_p \) and \( k \) are no longer constant but become space-dependent. Thus, we use \( \rho_0 f(x_i), \lambda_0 f(x_i), \mu_0 f(x_i), \beta_0 f(x_i), c_{p0} f(x_i) \) and \( k_0 f(x_i) \) to replace, respectively, in which \( \rho_0, \lambda_0, \mu_0, \beta_0, c_{p0} \) and \( k_0 \) are assumed to be constants and \( f(x_i) \) is a given non-dimensional function of space variable \( x_i \) \((z = 1, 2, 3)\).

We now considered a semi-infinite solid composed of FGMs, whose boundaries are traction free and keep the uniform temperature \( T_0 \) initially. For time \( t = 0 \) the surface of boundary \( x = 0 \) is suddenly raised to a constant temperature \( T_{\text{new}} \).

From the physics of the problem, it is clear that all the physical quantities will depend on \( x \) and \( t \) only. Thus, the displacement vector has the components:

\[
u_i = u(x, t), \quad u_t = u_0 = 0.
\]

Substituting these displacement vectors (4) into above equations in order and assuming \( f(x) = e^{\alpha x} \), we have

\[
\sigma_{ij} = e^{\alpha x} \left[ (\lambda_0 + 2\mu_0) u_{ij} - \beta_0 \theta \right],
\]

\[
\rho_0 \ddot{u}_i = (\lambda_0 + 2\mu_0) u_{ij} - \beta_0 \theta + \kappa \left[ (\lambda_0 + 2\mu_0) u_{ij} - \beta_0 \theta \right],
\]

\[
T_{sij} + \kappa T_{ij} = a_0 \left( \tau_{ij} \tilde{T} + T_0 \beta \tau_{ij} \ddot{x} + \ddot{\gamma}_{ij} \right).
\]

where \( \kappa \) is a non-dimensional constant indicating the non-homogeneity of FGMs, \( a_0 = \rho_0 c_{p0}/k_0 \) is the thermal viscosity constant.
Asymptotic Solutions of the Problem

General Solution in the Physical Domain

For simplicity, some following non-dimensional variables are introduced as follows

\[ x^* = a_0 v_x x, \quad t^* = a_0 v_x t, \quad \tau_0^* = a_0 v_x \tau_0, \quad k^* = k/a_0 v_x, \quad u^* = a_0 v_x (\lambda_0 + 2 \mu_0) u/\beta_0 T_0, \quad \theta^* = \theta/T_0, \quad \sigma_{ss}^* = \sigma_{ss}/\beta_0 T_0. \]

Substituting these non-dimensional variables into above equations (5)-(7) and dropping the asterisks for convenience, we have

\[ \sigma_{ss} = e^{\rho^*} (u_s - \theta), \]

\[ \ddot{u} = u_s - \theta + \kappa (u_s - \theta), \]

\[ \theta_{ss} + \kappa \theta_s = \tau_s \ddot{\theta} + \theta \theta_s = \tau_s \dd\theta_s + \theta \theta_s = \tau_s \dd\theta_s + \theta \theta_s, \]

where \( v_x = \sqrt{(\lambda_0 + 2 \mu_0)/\rho_0} \) is the speed of thermal elastic wave, and \( \theta = T_0 \beta_0^2 / a_0 k_0 (\lambda_0 + 2 \mu_0) \) is the thermoelastic coupling constant.

Applying the Laplace transform for the both sides of Eqs. (8)-(10), then we have

\[ \bar{\sigma}_{ss} = e^{\rho^*} (\bar{u}_s - \bar{\theta}), \]

\[ s^2 \bar{u} = \bar{u}_s - \bar{\theta} + \kappa (\bar{u}_s - \bar{\theta}), \]

\[ \bar{\theta}_{ss} + \kappa \bar{\theta}_s = \tau_s \bar{\theta} + \theta \bar{\theta}_s = \tau_s \bar{\theta} + \theta \bar{\theta}_s, \]

Eliminating term \( \bar{u} \) and \( \bar{\theta} \) separately by combining Eq. (12) and Eq. (13) results in

\[ \frac{d^2 \phi}{dx^2} + 2 \kappa \frac{d \phi}{dx} + \left[ \kappa^2 - s^2 - (1 + \theta) (\tau_0 s^2 + s) \right] \frac{d \phi}{dx} = 0, \]

where \( \phi (i=1,2) \) represent term \( \bar{u} \) and \( \bar{\theta} \), respectively.

The general solutions of Eq.(14) can be expressed as

\[ \phi = A_i (s) \exp (R_i x) + B_i (s) \exp (R_i x) + C_i (s) \exp (-R_i x) + D_i (s) \exp (-R_i x), \]

where \( R_i \) are the roots of following characteristic equation

\[ R^2 + 2 \kappa R + \left[ \kappa^2 - s^2 - (1 + \theta) (\tau_0 s^2 + s) \right] R + \theta (\tau_0 s^2 + s) = 0, \quad A_i (s), \quad B_i (s), \quad C_i (s) \quad \text{and} \quad D_i (s) \]

are coefficients depending on parameter \( s \) and are determined by the given boundary conditions.

Here the following non-dimensional boundary conditions on the boundary plane \( x = 0 \) are introduced

\[ \theta (0,t) = \theta_0 H (t), \quad \sigma_{ss} (0,t) = 0, \]

where \( H (t) \) is the Heaviside unit function and \( \theta_0 = (T_1 - T_0)/T_0 \) is a non-dimensional constant.

Applying the Laplace transform to the above condition (16), we have

\[ \bar{\theta} (0,s) = \theta_0/s, \quad \bar{\sigma}_{ss} (0,s) = 0. \]

Substituting these boundary conditions in general solutions (15) and stress component expression (11), and considering the bounded solutions with large \( x \) for an unbounded problem, the general solutions for \( u, \theta \) and \( \sigma_{ss} \) in the physical domain can be obtained as

\[ \pi = \frac{(\kappa - R_1) \theta_0 \exp (-R_1 x)}{(R_2 - R_1) \kappa (R_2 + R_1) s} \frac{(\kappa - R_0) \theta_0 \exp (-R_0 x)}{(R_2 - R_0) \kappa (R_2 + R_0) s}, \]

\[ \sigma_{ss} = e^{\rho^*} (u_s - \theta), \]

\[ \ddot{u} = u_s - \theta + \kappa (u_s - \theta), \]

\[ \theta_{ss} + \kappa \theta_s = \tau_s \dd\theta_s + \theta \theta_s = \tau_s \dd\theta_s + \theta \theta_s, \]
\[
\bar{\vartheta} = \frac{(\mathcal{R}_1^2 - \kappa R_1^2) \theta \exp(-R_1 x)}{(R_1 - R_2) \left( R_1 - \kappa (R_1 + R_2) \right)} + \frac{(\mathcal{R}_2^2 - \kappa R_2^2) \theta \exp(-R_2 x)}{(R_2 - R_1) \left( R_2 - \kappa (R_2 + R_1) \right)}.
\]

(19)

\[
\bar{\sigma}_{i} = \frac{s \theta \exp(kx) \exp(-R_1 x)}{(R_2 - R_1) \left( R_2 - \kappa (R_2 + R_1) \right)} - \frac{s \theta \exp(kx) \exp(-R_2 x)}{(R_2 - R_1) \left( R_2 - \kappa (R_2 + R_1) \right)}.
\]

(20)

**General Solutions in the Time Domain**

Since the complicated expressions of roots \( R \) \((i=1,2)\) contained in these transform solutions, some approximate treatment for roots \( R \) has been conducted by the asymptotic approach \([13-15]\), where the limit theorem of Laplace transform is used to obtain the following expressions:

\[
R_{i} = k_{i,2} s + m_{i,2},
\]

(21)

where \( k_{i,2} = \left[ 1 + \frac{\tau_0 + \vartheta \tau_0 + \sqrt{a_i}}{2} \right]^{1/2} \), \( m_{i,2} = \frac{1 + \vartheta \sqrt{a_i}}{4k_{i,2}} - \frac{\kappa}{2} \), \( a_i = (1 + \tau_0 + \vartheta \tau_0)^2 - 4\tau_0 \), and \( b_i = (1 + \vartheta) \tau_0 + \vartheta - 1 \).

Substituting these approximations (21) into transform solutions (18)-(20), the forms are convenient to inverse Laplace transform can be obtained. By means of the standard results of the Laplace transform, the corresponding asymptotic solutions in the time domain can be obtained as

\[
u = -\frac{k_1 \theta}{p} \exp(-m_1 x)(t-k_1 x)H(t-k_1 x) + \frac{k_2 \theta}{p} \exp(-m_2 x)(t-k_2 x)H(t-k_2 x),
\]

(22)

\[
\theta = \frac{\theta}{p} \exp(-m_1 x) \left[ k_1^2 - 1 + \left( 2\kappa k_1 - \frac{q}{p} (k_1^2 - 1) \right) (t-k_1 x) \right] H(t-k_1 x) - \frac{\theta}{p} \exp(-m_2 x) \left[ k_2^2 - 1 + \left( 2\kappa k_2 - \frac{q}{p} (k_2^2 - 1) \right) (t-k_2 x) \right] H(t-k_2 x),
\]

(23)

\[
\sigma_{ii} = \exp(k x) \left( \frac{\theta}{p} \left[ \exp(-m_1 x) \left[ 1 - \frac{q}{p} (t-k_1 x) \right] H(t-k_1 x) - \exp(-m_2 x) \left[ 1 - \frac{q}{p} (t-k_2 x) \right] H(t-k_2 x) \right] \right),
\]

(24)

where \( p = k_1^2 - k_2^2 \), \( q = \kappa (k_2 - k_1) + 2(m_1 k_2 - m_2 k_1) \).

**Asymptotic Solutions of the Problem**

**Wave Propagation Analysis**

Due to the properties of Heaviside unit function, two waves, named as thermoelastic wave and thermal wave respectively, would generate from the boundary, whose propagation velocities and positions of wavefront can be derived as

\[
v_{i,2} = \frac{1}{k_{i,2}}, \quad \xi_{i,2} = \frac{t}{k_{i,2}}.
\]

(25)

Combining with the expressions of parameters \( k_{i,2} \), we can observe that both propagation velocities \( v_{i,2} \) and positions \( \xi_{i,2} \) of each wavefront are dependent on relaxation time \( \tau_0 \) and thermoelastic coupling constant \( \vartheta \). The distributions of propagation velocities \( v_{i,2} \) versus relaxation time \( \tau_0 \) at different values of \( \vartheta \) are shown in Fig.1.

We can clearly observe that \( v_{i,2} \) are decreasing with increasing, which indicates the propagation of two waves would be difficult with the enhancement of the delay effect. On the other hand, the effects of \( \vartheta \) on the propagation of two waves are different, \( v_1 \) is decreasing with the increase of \( \vartheta \), but is increased for \( v_2 \). Furthermore, it is noted that the non-homogeneous parameter \( \kappa \) is not contained in expressions \( k_{i,2} \), which means that the non-homogeneity of FGMs has no effect on the propagation of two waves.
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**Thermoelastic Response Analysis**

Now for the purpose of illustration, we consider copper like material with material constants [9, 16], and some values of non-dimensional parameters are taken as $\tau_0 = 0.5$, $\vartheta = 0.0168$, $\theta_0 = 1.0$.

Figures 2-4 display the distributions of displacement $u$, temperature $\theta$ and stress component $\sigma_{xx}$ for a wide range of $x$ ($0 < x < 2.5$) at different time $t$ and given value of $\kappa = -0.5$. The important phenomenon for generalized thermoelastic problem, that is, all of $u$, $\theta$ and $\sigma_{xx}$ vanish identically at all positions beyond the faster wavefront, can be observed clearly from these distributions. The displacement is continuous at all positions including each wavefront, but the temperature and stress are discontinuous at each wavefront, and two jumps would be generated in each wavefront.

Figures 5-7 displays the variation of each distribution for different values of non-homogeneous parameter $\kappa$ at given time $t = 0.5$, where $\kappa = 0.0$ corresponds the homogeneous case with constant material properties. Obviously the non-homogeneity has a significant effect to each distribution. The magnitudes of displacement $u$, temperature $\theta$ and stress component $\sigma_{xx}$ are decreasing with the increasing of parameter for the case of $\kappa < 0$, but all the magnitudes are increasing for the case of $\kappa > 0$, which means the effect of non-homogeneity is dependent on the real variation of material properties with the spatial positions. Furthermore, it is noted that the variation of temperature and stress is more significant than that of displacement for the different values of parameter $\kappa$, which means the temperature and stress have the stronger dependency on non-homogeneity of FGMs.
FIGURE 6. DISTRIBUTION OF TEMPERATURE FOR $t = 0.5$

FIGURE 7. DISTRIBUTION OF STRESS COMPONENT FOR $t = 0.5$

Conclusions

The thermoelastic response in a functionally graded solid is studied by an analytical method in this paper. The analysis of the results permits some concluding remarks:

1) The displacement, temperature and stresses have a delay distribution when the heat signal propagates with a finite speed. The propagations of each wave are dependent on the relaxation time $\tau_0$ and thermoelastic coupling constant $\Theta$, but are independent on non-homogeneous index $\kappa$.

2) The presence of non-homogeneous parameter has significant effect on each distributions. Comparison with the effect on displacement, temperature and stresses are more sensitive to the variation of material properties.

REFERENCES