Modeling of Large Deformations of Hyperelastic Materials

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Abstract
The elastomer properties (large deformations, damping) make their use more and more common in industries such as aerospace, automotive, construction and civil engineering or even the entertainment industry... Modeling the behavior of such materials is highly nonlinear, the nonlinearities are both geometric (due to large deformations imposed) and behavioral (behavior laws used are nonlinear). This paper presents a detailed description of the numerical implementation of incompressible isotropic hyperelastic behavior. In this study, the analysis of large deformation problems of Ogden’s hyperelastic is based on the finite element method (FEM) and mathematical programming. To solve the problem of balance, we suggest using the combined Newton-Raphson/Arc-length procedure. A typical example is presented to illustrate the performance of this formulation.

Keywords
Elastomers; Hyperelasticity; Large Strain; FEM; Newton-Raphson Algorithm; Arc-Length Scheme

Introduction
The theoretical studies of hyperelastic materials have mobilized many researchers for decades; see eg the work of Treloar [1], Gent [2] and Frakley [3]. However, the first important works were proposed by Mooney [4] and Rivlin [5]. Valani and al. [6] proposed to write the strain energy in a form separated following the main directions. This led to the widely used model of Ogden [7, 8]. Recently, different models have been developed for hyperelastic materials especially in biomechanics (Zulliger and al. [9]).

The objective of this paper is to present the model most widely used hyperelastic behavior and in particular the isotropic model. To do this, the different results of basic mechanics of the major changes are outlined in the first place. Are recalled and the various tensor quantities used in the modeling of continuous medium and medium formulation based on isotropic hyperelastic tensor structure. In a second step, the most widely used hyperelastic potential is exposed.

Description of Large Strains
Elastomers, with hyperelastic behavior, usually work in large strains. Note \( \vec{x} \) and \( \vec{X} \) positions vectors of a particle \( P \) of a deformable body in the current configuration (where the solid occupies the volume \( \Omega \)) and initial (where the solid occupies the volume \( \Omega_0 \)).

For a Lagrangian description, the movement of the body can be defined relative to a reference configuration \( C_0 \), by a vector function:

\[
\Phi : (t) \rightarrow x = \Phi(X, t).
\]

(1)

By introducing the displacement vector \( \vec{u} \) and we write the equation (1) in the equivalent form:

\[
\vec{x} = \vec{X} + \vec{u}(\vec{X}, t).
\]

(2)

Order to describe the geometric transformations associated with these large strains, we introduce the deformation gradient tensor \( F \) defined by:
\[ F_{\gamma} (\hat{X}, t) = \frac{\partial \Phi (\hat{X}, t)}{\partial X_j} = \frac{\partial x}{\partial X_j} = \delta_{ij} + \frac{\partial u_i}{\partial X_j}, \quad (3) \]

Where
\[ \delta_{ij} \]
denotes the Kronecker symbol.

Or equation (3) can be written in matrix form:
\[ F = I + \nabla u, \quad (4) \]

Where
\[ I \]
is the unit tensor,
\[ \nabla \]
is the gradient tensor of displacements.

Due to the large displacements and large rotations, the strain tensor of Green-Lagrange \( E \) was adopted to describe the nonlinear relationship between the deformation and displacement:
\[ E = \frac{1}{2} (C - I), \quad (5) \]

Where
\[ C = F^T F \]
is the strain tensor of Cauchy-Green right.

In the case of a hyperelastic law, there is a strain energy density \( W \) which depends on one of the strain tensor and whose derivative gives the second stress tensor of Piola-Kirchhoff \( S \):
\[ S_{ij} = \frac{\partial W}{\partial E_{ij}} = 2 \frac{\partial W}{\partial C_{ij}}. \quad (6) \]

In the particular case of isotropic hyperelasticity \[10\], equation (6) can be written as follows:
\[ S = 2 \left[ I_3 \frac{\partial W}{\partial I_3} C^{-1} + \left( \frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2} \right) - \frac{\partial W}{\partial I_2} C \right], \quad (7) \]

Where
\[ I_i \quad (i = 1, 2, 3) \]
are the invariants of the Cauchy-Green tensor on the right, \( C \), such as:
\[ I_i = tr(C) ; \quad I_3 = \frac{1}{2} \left[ I_1^2 - tr(C^2) \right] \text{ and } I_3 = \det(C), \quad (8) \]

Where
\( tr(\bullet) \) and \( \det(\bullet) \)
depict, respectively, the trace and the determinant of a tensor.

Note that \( S \) has no physical meaning but it is symmetric and purely Lagrangian, and from which one can determine the expression of the Cauchy tensor \( \sigma \) :
\[ \sigma = \frac{1}{\det(F)} F S F^T. \quad (9) \]

For the Ogden model \[7\] for incompressible materials, the energy density is expressed in terms of eigenvalues \( \lambda_j \) \( (j = 1, 2, 3) \) of the Cauchy-Green tensor right by:
\[ W(\lambda_1, \lambda_2, \lambda_3) = \sum_k \frac{\mu_k}{\alpha_k^m} \left( \lambda_1^{\alpha_k} + \lambda_2^{\alpha_k} + \lambda_3^{\alpha_k} - 3 \right), \quad (10) \]

where the constants are the material parameters and \( m \) is the total strain energy terms.

We recall that the incompressibility of the material results in \( \det(F) = 1 \) on \( \lambda_1 \lambda_2 \lambda_3 = 1 \). Hence, the energy density can be expressed only by \( \lambda_1 \) and \( \lambda_2 \):
\[ W(\lambda_1, \lambda_2, \lambda_3) = \sum_k \frac{\mu_k}{\alpha_k^m} \left( \lambda_1^{\alpha_k} + \lambda_2^{\alpha_k} + \frac{1}{\lambda_1^{\alpha_k} \lambda_2^{\alpha_k}} - 3 \right). \quad (11) \]

**Variational Formulation**

Solving the problem balance of incompressible hyperelastic is reduced to problem of minimizing the functional \( \Psi \):
\[
\begin{aligned}
\text{Min} \quad & \Psi (u, S) = \int_{\Omega} (S.E(u) - \bar{f}.u) d\Omega - \int_{\Gamma_t} \bar{T}.u d\Gamma \\
\text{Subject to} \quad & \text{Conditions of incompressibility}, \\
& \text{Boundary conditions} (u = \bar{u} \text{ on } \Gamma_u).
\end{aligned} \quad (12)
\]

Where
\( \Omega \) is an isotropic structure, subject to surface forces \( \bar{\tau} \) on the surface \( \Gamma_t \),
\( \bar{f} \) is the gravity forces,
\( \bar{u} \) is displacements imposed on the surface \( \Gamma_u \).

The nonlinear equations (12) are solved numerically with the finite element method whose the approximation of the displacement field is defined by the relation:
\[ u(x) = N(x)U \text{ with } \varepsilon = \nabla (N(x)U), \quad (13) \]

Where
\( x = x, y, z \) are the nodal coordinates,
\( N(x) \) is the shape functions matrix,
\( U \) is the vector of unknown nodal displacements.

Therefore, the functional \( \Psi \) takes the discretized form as follows:
Thus, the discrete system (14) can be solved iteratively by using the combined Newton-Raphson/Arc-length procedure (see [11] for details on the implementation of this method in the context finite element).

**Numerical Example**

In this example, we consider a plate with a central circular hole in the stress plane and tension is sought under a constant displacement up to 100% elongation. Geometry (L × D = 25.4 × 14.24 cm²) with a unit thickness is shown in Fig. 2. For more details see reference [12]. Due to symmetry of geometry, only one quarter of the plate is modeled, as shown below.

**Conclusions**

A computational method by using the mathematical programming and the analysis approach hyperelastic is developed here to investigate the large deformation. A finite element model in conjunction with the combined Newton-Raphson/Arc-length procedure was adopted allowing investigating interesting applications. The obtained results illustrate the effectiveness of our approach developed.

This work can be extended in the future by taking into account other parameters as frictional contact with heat transfer.

**REFERENCES**


El Hassan Boudaia is an associate Professor at the Mechanical Engineering Department of Faculty of Science and Technology of Beni Mellal, Morocco. His Ph.D thesis in 2007 was on the subject of a meshless and finite element methods analysis for elasto-plastic contact problems with friction. His current research activities are multidisciplinary and focus mainly on design approaches and reliability analysis, especially in structural engineering. Using the elasto-plastic analysis and the meshless or the finite element method, he has published with his coworkers many works on the reliability assessment of structures, especially in the non-associated elasto-plasticity area.