Structural synthesis of over-constrained parallel mechanisms based on screw theory

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Abstract: In this paper, the theory of screw is briefly discussed. According to the different result of platform’s moving situation caused by constraints’ linear association, the linear association of the platform constraints are classified into two types. The one may cause platform singularity in the design configuration and destroy the rightness of the design, it needs to be avoided. On the opposite side, linear associated constraints may form a new over-constrained parallel manipulator, it needs to be used. We present a method for avoiding platform singularities by avoiding constraints’ linear association. And we also present the structuring method for over-constrained parallel manipulators by using constraints’ linear association. Finally, some design sketches are shown as examples.

Keywords: structural synthesis; parallel mechanism; singularity; constraint; screw theory

1. Introduction

A parallel manipulator typically consists of a moving platform that is connected to a fixed base by several limbs. The number of limbs is usually equal to the number of degrees of freedom (DoF). Parallel manipulators have attracted much attention among researchers in the past two decades because they have the advantages of higher stiffness, more accuracy, and higher load carrying capacity as compared with the conventional serial manipulators. Initially, most of designs are Stewart parallel mechanism and its extensive forms with 6-DoF. However, a 6-DoF fully parallel manipulator has the limitations of small workspace volume, difficult mechanical design and more difficult kinematics, dynamics and control because of its excessive number of limbs. Therefore, fewer than 6-DoF, or called low-DoF, parallel manipulators have been investigated to overcome the shortcomings of 6-DoF fully parallel manipulators.

Firstly, researchers designed such mechanisms by their talent idea and experience from long time studying. Finally, some systematic methodologies were presented for structural synthesis. Researchers have done great work in this field[5,6,7]. In general, these methodologies can be classified into two types. One is based on the theory of screw[2,3,4], and another is based on the Li group. Relatively, the method based on screw theory is more successful one[12,15,16]. In the whole view, the structural synthesis approach based on the theory of screw is clear and simple[11,13,14]. Because any moving body in space has 6 DoFs at most, while a part of freedoms are constrained by relative constraints, the body’s freedom will be decreased consequently. On the other side, forces and couples provided by limbs acting on the platform can be expressed by screw forms. If any virtual work produced by the two screw systems that the one expresses platform moving, and another expresses such constraints acting on the platform is equal to zero, the desired degree of freedom is left. It means if the order of constraints’ screw is n, the relative freedom of platform is 6-n.

Since there is a problem while the constraints are linear associated, the order of constraints’ screw will be decreased to a number less than n, at the same time, the freedom number of platform will be more than 6-n. Commonly, the singularities are caused and the moving characteristic is destroyed by this reason[8,10]. Because of the increased but undesired degree freedoms, the whole mechanism will be in an uncontrollable situation. However, this kind of constraints linear association sometimes has its advantages for structuring some over-constrained parallel manipulators[1,9]. In this situation, two or more constraints are linear associated and they over constrain the platform, in another word, two or more constraints acting on the platform just like one of them acting on the platform in the view of constraint. The reason of more constraints is for having enough limbs to equal the freedoms degrees of the platform, therefore, only one actuator will be mounted on the each limb to keep a good dynamic characteristic.

Some researchers have studied the singularity problem for current exiting parallel manipulators to avoid its shortcomings. However, how to avoid singularity caused by constraint’s linear association in the process of structural synthesis is still an open problem. Here, we call it singularity in the design configuration, and it is caused by constraints linear association in the design configuration. In the opposition situation, the advantage of the constraints linear association is that most of over-constrained parallel manipulators are designed based on this situation. But no one discussed that how to use its advantages for structuring over constrained parallel manipulators in system. Therefore, it’s very important and necessary issue to analyze these two different cases.

In this paper, firstly, the theory of screw is briefly discussed. Then, we classify constraints’ linear association into two types initially, between them, the one needs to be avoided, and another needs to be used. And we present a method for avoiding platform singularities by avoiding constraints’ linear association. Furthermore, we also present the structuring method for over constrained parallel manipulators by using constraints’ linear association too. Finally, some design sketches are shown as examples.

2. The theory of screw

In this section, the definition of screw and reciprocal screw are reviewed. A unit screw, \( \mathbf{s} \), is denoted by a dual vector \( (\mathbf{s}, \mathbf{s}^0) \). If \( \mathbf{s} \cdot \mathbf{s} = 1 \) were satisfied, the screw is called the unit screw. The pitch of the screw \( \mathbf{h} \) is defined as:

\[
\mathbf{h} = \frac{\mathbf{s} \cdot \mathbf{s}^0}{\mathbf{s} \cdot \mathbf{s}}
\]

(1)

If \( \mathbf{h} = 0 \), namely, \( \mathbf{s} \cdot \mathbf{s}^0 = 0 \), we call the screw \( \mathbf{s} \) a line vector, denoted by \( (\mathbf{s}, \mathbf{s}^0) \), where \( \mathbf{s} \) is a vector. \( \mathbf{s}^0 = \mathbf{r} \times \mathbf{s} \) defines the moment of the screw axis about the origin of a reference frame, \( \mathbf{r} \) is the position vector of any point on the vector \( \mathbf{s} \) with respect to the reference frame. If \( \mathbf{h} \) were equal to infinite, namely, the screw’s form is \( (0, \mathbf{s}) \), an infinite pitch screw can represent a linear velocity in kinematics or a force moment in static. If two screws, \( \mathbf{s} \) and \( \mathbf{s'} \), satisfy the following condition

\[
\mathbf{s} \circ \mathbf{s'} = 0
\]

(2)
They are said to be reciprocal, and the symbol \( \ast \) denotes the reciprocal product. If the screw \( S \) represents an instantaneous motion of a rigid body, the first three components of it represent the angular velocity and the last three components represent the linear velocity of a point in the rigid body that is instantaneously coincident with the origin of a reference frame, and we call it a twist. In comparison, the reciprocal screw \( S_r \) denotes a system of forces and moments acting on a rigid body, then the first three components of it represent the resultant force and the last three components represent the resultant moment about the origin of a reference frame, and we call it a wrench. A wrench represents a single force if the pitch is equal to zero, and a single force moment if the pitch is infinite.

3. Freedoms and constraints analysis

All reciprocal screws provided by limbs form a constraint system. Assume each symmetrical limb has \( m \)-order motion screws, and the platform’s freedom degree is \( n \leq m \), and the limb’s number is \( k \leq n \). While \( k = n \), the number of limbs is equal to the number of freedom degrees, this is the most desired type to keep better dynamic characteristic. So, the total constraint number is \( k(6 - m) \), and the following expression must be satisfied to keep right design.

\[
\text{Rank} \left[ S_r^1 \cdots S_r^{k(6-m)} \right] = n
\]  

(3)

As an example, Figure 1 shows the relationship between constraint screws and motion screws provided by one limb with 5 single-DoF pairs, those constraints both acting to the platform and limb at the same time. It shows one constraint screw \( S_r \), and 5-order motion screws \( S_1, S_2, S_3, S_4, S_5 \), and these screws form a 6 orders screw system. If the limb has 4 or 3 single DoF pairs, the number of constraint screw is 2 or 3 respectively.

![Fig. 1. Setting for document template.](image)

Here, all possible parallel manipulators in view of the number of limbs, number of constraints, and number of limbs screw orders are enumerated as listed in table 1. In Table 1, only in the situation that marked by \( \sqrt{\text{√}} \), the number of total constraints adds on the number of platform’s DoF is 6. In this situation, if all constraint screws provided by each limb are non-linear associated, the moving characteristic will be kept well. Furthermore, all limbs’ constraints form a system that decides moving characteristic of the parallel manipulator together. Only one limb can not perform the constraints task, and the resultant of constraints provided by all limbs will decide the final DoF of the parallel manipulator. So, if they are linear associated, the constraint screws’ order will degenerate, and increase one or more extra DoF of platform and cause the singularity problems. But in the other situations, the number of total constraints adds on the number of platform’s DoF is more than 6, i.e., the number of constraints is more than its basic need. Therefore, if they are not linear associated, the platform will have some extra constraints and decrease the designed DoF. In this situation, constraints' linear association is useful for structuring over constrained parallel manipulators. So, we have to study how to classify these two types of constraints’ linear association situation and use them in structuring process respectively.

<table>
<thead>
<tr>
<th>DoF of PM</th>
<th>Number of links</th>
<th>Link’s screw orders</th>
<th>Number of each limb’s constraint</th>
<th>The total number of constraints</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>12</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>3</td>
<td>1</td>
<td>6</td>
</tr>
</tbody>
</table>

Table 1 The constraints and screw number of PM

4. Method avoiding platform singularities

In this section, we present how to avoid constraints linear association to avoid the singularity in the design configuration. Taking 3-DoF translational parallel manipulators as an example, we build a family of parallel manipulators that can satisfy the design requirements but
without singularity in the design configuration. According to above analysis, to structure a family of 3-DoF translational parallel manipulators, the 3-left rotation freedom of the platform must be constrained by 3 independent limbs. Three constraints can be written in the screw form as:

\[ S_{ri} = [0,0,0; l_i, m_i, n_i] \quad (i = 1,2,3) \]  

(4)

In view of constraints’ linear association or not and its effect, there are 2 kinds of cases.

Case 1: Any of limbs can perform the total constraints task for moving platform, thus

\[ l_i \neq 0, m_i \neq 0, n_i \neq 0, \quad (i = 1,2,3) \]  

(5)

Resume limb-1 can satisfy this demand,1 the following two equations will be met at the same time or respectively when the constraints are linear associated:

\[ S_{r2} = [l_2, m_2, n_2] = k_1 S_{r1} = k_1 [l_1, m_1, n_1] \]  

(6)

\[ S_{r3} = [l_3, m_3, n_3] = k_2 S_{r1} = k_2 [l_1, m_1, n_1] \]  

(7)

In this case, two or one of 3 limbs will lose function. Therefore, the platform will lose stability. Actually, it loses the advantage of parallel manipulators and fall into the serial mechanism. So, we have to avoid this case.

Case 2: When the one or two parameters of \( l_i, m_i, n_i \) is zero, any one limb of 3 limbs can not perform the constraints task for moving platform. It needs 3 or at least 2 limbs to form a resultant effect to decide the DoF of parallel manipulators. Actually, this is a necessary condition of parallel mechanism. When the constraints are linear associated, the order of platform constraints will degenerate and may strongly destroy the kinematic and dynamic characteristic of parallel manipulators. So, to find 3 1-order linear independent of constraint system to form an orthogonal constraint system is a good choice to guarantee free-singularities in design configurations. Firstly, we decompose the constraint screw into several 1-DoF screws. Then, we make all 1-order screws in spatial orthogonal. Finally, we use the reciprocal equation to get all kinds of limb’s structures.

According to this method, we obtain 3 linear independent constraints for a 3-DoF translational parallel manipulator as follows.

\[ S_{r1} = [0,0,0;1,0,0] \quad S_{r2} = [0,0,0;0,1,0] \quad S_{r3} = [0,0,0;0,0,1] \]

And we obtain basic pairs’ screw system of limb as follows:

\[ S_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \]  

(8)

\[ S_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \]  

(9)

\[ S_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \]  

(10)

Since any basic 1-order pair can be expressed by a screw as equation (2) described, by linear combining them as any possibilities, we can get all possible structures of 3-DoF translational parallel manipulators. For example, figure 2 shows a kind of parallel manipulator that is singularity free in the design configuration as. In this parallel manipulator, each of 3 limbs has 5 1-order revolute pairs, among these 5 pairs, their axes belong to two group of parallel lines. The first 3 revolute pairs’ axis from base are parallel to each other, and the last 2 revolute pairs’ axis are parallel to each other. Furthermore, two group of axes in the same limb of 3 limbs are parallel to the plane XOY, XOZ, YOZ respectively to keep their constraint in orthogonal.
5. The situation for using constraints linear associated

In section 3, we discuss how to structure parallel manipulator without singularity. Designers have to arrange constraints specially to avoid linear association. On the opposite situation, designers must arrange constraints specially to make them linear association. Traditionally, 4-DoF and 5-DoF parallel manipulators should have 4 and 5 limbs to make sure each limb with unique actuator. But the moving platform only needs 2 or 1 constraints. Whatever the limbs are arranged, they must be linear dependent to make sure the resultant constraint is 2-order or 1-order. Thus, keeping linear dependence is a necessary step for the structural synthesis. For 4-DoF and 5-DoF parallel manipulators with 4 and 5 limbs respectively, each limb provides a screw constraint as $S_{i\mu}, i = 1, 2, 3, 4$ (for 4-DoF) and $S_{i\nu}, i = 1, 2, 3, 4, 5$ (for 5-DoF), to keep the right design of constraint acting on the platform. All constraint should satisfy the following equation:

$$\text{Rank}\left[ S_{1\mu}, S_{2\mu}, S_{3\mu}, S_{4\mu} \right] = 4 \quad \text{(For 4-DoF)}$$

$$\text{Rank}\left[ S_{1\nu}, S_{2\nu}, S_{3\nu}, S_{4\nu}, S_{5\nu} \right] = 5 \quad \text{(For 5-DoF)}$$

Here, we take a 4-DoF parallel manipulator without translation mobility along $x$-axis and $y$-axis (Fig. 3) to explain that making constraint screws’ linear association is necessary. It has 4 symmetrical limbs. Among these 4 limbs, any 2 neighbor limbs are assembled in two orthogonal planes, and any two opposite limbs are assembled in two parallel planes. Each limb has 4 pairs. Three of them are 1-order rotation pair, and one of them is 2-order cylindrical pair as the third pair. All the third and the fourth pairs in all limbs intersect into the same point. The first and second rotation pair in each limb are in parallel.

For analyzing its DoF and each limb’s moving screw system, we structure a fixed coordinate system as Figure 3 shows. Assume the rotation center point is $O(x, y, z)$, $r$ is the vector of $O_1O$, and $S_{ij}, (i = 1, 2, 3, 4; j = 1, 2, 3, 4, 5)$, represents the vector of jth pair’s axis of the ith limb. we have:

(1) $S_{1\mu}$ is parallel to $S_{i\mu}$. $S_{1\nu}$ is in the same plane, $S_{1\nu}$ is parallel to $S_{3\nu}$, and $S_{2\nu}$ is parallel to $S_{4\nu}$.

(2) $S_{1\mu}$, $S_{1\nu}$ and $S_{1\mu}$ intersect into the center point. $O(x, y, z)$, and $S_{1\mu}$ and $S_{1\nu}$ is orthogonal to $S_{1\mu}$.

(3) $(L_y, M_y, N_y)$ and $(P_y, Q_y, R_y)$ represent the vector of pair axis’ direction and position of $S_{ij}$, respectively.

(4) $S_{1\mu}$ is through into original point, so, $r_{1\mu} = 0$.

(5) $S_{1\mu}$ is orthogonal to $S_{1\nu}$, so two vector’s product is zero, and we have, $P_{1\mu} = L_{1\mu} = 0$.

(6) $S_{1\mu}$ and $S_{1\nu}$ are through into original point, and we have, $r_{1\mu} = r_{1\nu} = r$. 

Fig. 2. 5R singularity free parallel.

Fig. 3. Setting for document template.
After obtaining their geometrical and dimension relationship, we can describe each limb’s motion in the screw system form \( S_1, S_2, S_3, S_4 \) as follows:

\[
S_1 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & Q_{12} \\
L_{13} & M_{13} & N_{13} & yN_{13} - zM_{13} & zL_{13} - xN_{13} & xM_{13} - yL_{13} \\
0 & 0 & 0 & L_{13} & M_{13} & N_{13} \\
L_{15} & M_{15} & N_{15} & yN_{15} - zM_{15} & zL_{15} - xN_{15} & xM_{15} - yL_{15}
\end{bmatrix}
\]

\begin{equation}
(13)
\end{equation}

\[
S_2 = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & R_{23} \\
L_{23} & 0 & N_{23} & yN_{23} & zL_{23} - xN_{23} & yL_{23} \\
0 & 0 & 0 & P_{23} & 0 & R_{23} \\
L_{25} & M_{25} & N_{25} & yN_{25} - zM_{25} & zL_{25} - xN_{25} & xM_{25} - yL_{25}
\end{bmatrix}
\]

\begin{equation}
(14)
\end{equation}

\[
S_3 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & P_{32} & Q_{32} \\
0 & 0 & M_{33} & N_{33} & yN_{33} - zM_{33} & -xN_{33} \\
0 & 0 & 0 & 0 & Q_{34} & R_{34} \\
L_{35} & M_{35} & N_{35} & yN_{35} - zM_{35} & zL_{35} - xN_{35} & xM_{35} - yL_{35}
\end{bmatrix}
\]

\begin{equation}
(15)
\end{equation}

\[
S_4 = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & P_{42} & 0 \\
L_{43} & 0 & N_{43} & yN_{43} & zL_{43} - xN_{43} & yL_{43} \\
0 & 0 & 0 & 0 & Q_{44} & R_{44} \\
L_{45} & M_{45} & N_{45} & yN_{45} - zM_{45} & zL_{45} - xN_{45} & xM_{45} - yL_{45}
\end{bmatrix}
\]

\begin{equation}
(16)
\end{equation}

Based on the reciprocal theory and equation (2), we can get the reciprocal constraints of each limb as the following equations.

\[
S_{1p} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & z - y \end{bmatrix}, \quad S_{2p} = \begin{bmatrix} 0 & 1 & 0 & -z & 0 & x \end{bmatrix},
\]

\[
S_{3p} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & z - y \end{bmatrix}, \quad S_{4p} = \begin{bmatrix} 0 & 1 & 0 & -z & 0 & x \end{bmatrix}
\]

Till now, we get all constraints acting on the platform, and their resultant effect will decide the mobility degree number and its characteristic. Make these 4 screws as a matrix form, we can find that 4 constraints acting on the platform are linear associated. So, we get the rank of this constraints' matrix

\[
\text{Rank}\left[ S_{1p}, S_{2p}, S_{3p}, S_{4p} \right] = 2
\]

Consequently, the remaining DoF of platform is \( 6 - \text{Rank}\left[ S_{ip}, i = 1,2,3,4 \right] = 4 \). According to the above analysis, we can prove that a parallel manipulator has 4–DoF with 3 rotation degrees and 1 translation degree along z-axis. Generally, most of over-constrained parallel manipulators are structured by this method. So, in the structural synthesis process for such mechanisms, we usually make screw constraints in linear associated by arranging them in special geometrical or dimensional relationship. This method is opposite application type as compare to the application type of avoiding linear association for singularity free design process described in section 3. Based on the traditional spatial freedom formula, Kutzbach-Grubler formula, to calculate the freedom of this parallel manipulator, we have:

\[
F = 6(n - j - 1) + \sum_{i=1}^{f} f_{i} = 6 \times (14 - 16 - 1) + 20 = 2
\]

where, \( n \) is the number of total parts, \( j \) is the number of total pairs, \( f_{i} \) is the freedom of the \( i \)th pair, and \( F \) is the freedom of parallel manipulator. Actually, the freedom is four just like we analyze based on the theory of screw. In this parallel manipulator, we increase two limbs to keep it with four limbs. In fact, we increase two constraints to the platform. Because we arrange them in parallel and keep them linear associated, we get the right design by this way. At the same time, we violate the traditional freedom formula, and it is called over constrained. Otherwise, we can find if we keep all constraints non-linear associated. We just need two limbs to constrain the platform. By cutting two limbs from this parallel manipulator, we have the structure as Fig4. And the freedom can be calculated as:
\[ F = 6(n - j - 1) + \sum_{i=1}^{j} f_i = 6 \times (8 - 1) + 10 = 4 \]  

(19)

We find that the freedom number is right. However, only with two limbs it will degenerate into a link. Therefore, we use the constraint linear association to keep the right structural synthesis that can satisfy the design condition of parallel manipulators. And this is a different application method as compare to the avoiding linear association described in section 3.

6. Conclusions

In conclusion, we analyze the constraints linear dependence, into two types. 1) When the number of constraints is equal to the number of DoFs, we have to arrange the constraints to avoid linear association. 2) When the number of constraints is more than the 6-DoFs, we have to arrange them to keep linear association. Therefore, the redundant constraints may not decrease the desired DoF of platform. This provides a useful method in the process of structural synthesis both for designing singularity free in the design configuration and for over-constrained parallel manipulators.

References